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Infinity

INTRODUCTION

One of the persistent questions concerning infinity is whether we can know or comprehend it. Another is whether the infinite exists, and if so, to what kind of thing infinity belongs. It is not surprising, therefore, that the discussion of infinity often borders on the unintelligible.

The idea of infinity, like the idea of eternity, lacks the support of the imagination or of sense-experience. The fact that the infinite cannot be perceived or imagined seems sufficient to lead Hobbes and Berkeley to deny its reality. "Whatsoever we imagine is *finite*," writes Hobbes. "Therefore there is no idea, or conception of anything we call *infinite* . . . When we say anything is infinite, we signify only that we are not able to conceive the ends and bounds of the thing named, having no conception of the thing, but of our own inability."

On similar grounds Berkeley rejects the possibility of infinite division. "If I cannot perceive innumerable parts in any infinite extension," he writes, "it is certain that they are not contained in it: but it is evident, that I cannot distinguish innumerable parts in any particular line, surface, or solid, which I either perceive by sense, or figure to myself in my mind; wherefore I conclude that they are not contained in it."

But for most of the great writers on the subject, the impossibility of representing infinity and eternity to the imagination does not render them inconceivable or meaningless. Yet it does account for the difficulty of grasping their meaning, a difficulty further increased by the fact that, whatever their meaning, *infinity* and *eternity* are indefinable. To define the infinite would be to limit—even in thought—the unlimited.

The notion of infinity involves greater perplexities than that of eternity. The meaning of eternity is weighted with the mystery of God, the world, and time. All these affect the conception of infinity; but for the infinite there are also the mysteries of number and of space, of matter and motion. In the sphere of quantity, or of things subject to quantity, infinity is itself the source of mystery, or at least the root of difficulty in analysis. It is the central term in the discussion of the continuous and the indivisible, the nature of series and of limits.

As indicated in the chapter on Eternity, that idea in each of its applications seems to have one or the other of two meanings—(1) the meaning in which it signifies infinite time, time without beginning or end, and (2) the meaning in which it signifies the timelessness or immutability of being. Both meanings are negative, so far as our understanding is concerned. Yet what is signified by the second is in itself something positive, at least in the opinion of those who think that to be exempt from change entails having every perfection or being lacking in nothing.

This split in meaning also occurs in the idea of infinity. As applied to being, the term *infinite* signifies something positive, even though our understanding of what is signified remains negative or, at best, analogical. An infinite being is one which lacks no attribute that can belong to a being. This is the positive condition of absolute perfection. The infinite here still means the unlimited, but that which is unlimited in being has no defect. To lack deficiencies is to be perfect.

It is in this sense that Spinoza defines God as "Being absolutely infinite, that is to say, substance consisting of infinite attributes, each one of which expresses eternal and infinite essence." Like Spinoza, Aquinas maintains that "besides God nothing can be infinite." But he distinguishes the absolute or positive sense in which God alone is infinite from the sense of the word in which it can be said that "things other than God can be relatively infinite, but not absolutely infinite." This other meaning, according to Aquinas, is not only relative but negative, for it connotes "something imperfect." It signifies indeterminacy or lack of perfection in being.

What Aquinas calls the relative or potential infinite, he attributes to matter and to quantities—to bodies, to the magnitudes of space and time, and to number. This sense of "infinite" corresponds to that meaning of "eternal," according to which time consists of an endless series of moments, each having a predecessor, each a successor, no matter how far one counts them back into the past or ahead into the future.

But in the field of quantities other than time, the meanings of infinite and eternal part company. There is, of course, some parallelism between infinite space and infinite time, insofar as an infinite extension is one which does not begin at any point or end at any; but the consideration of space and number leads to an aspect of infinity which has no parallel in the consideration of eternity.

"In sizes or numbers," Pascal writes, "nature has set before man two marvelous infinities . . . For, from the fact that they can always be increased, it follows absolutely that they can always be decreased . . . If we can multiply a number up to 100,000 times, say, we can also take a hundred thousandth part of it by dividing it by the same number we multiply it with, and thus every term of increase will become a term of division by changing the integer into a fraction. So that infinite increase includes necessarily infinite division." As endless addition produces the infinitely large, so endless division produces the infinitesimal or the infinitely small.

A trillion trillion is a finite number, because the addition of a single unit creates a larger number. The fact that the addition of another unit produces a different number indicates that a trillion trillion has a determinate size, which is the same as saying that it is a finite number. An infinite number cannot be increased by addition, for it is constituted—in thought at least—as a number larger than the sum of any two finite numbers; which is another way of saying that it is approached by carrying on the process of addition endlessly.

What Galileo points out about two infinite quantities seems to hold for an infinite and a finite quantity. He asks us to consider the totality of all integers (which is infinite) and the totality of their squares (which is also infinite). On the one hand, there appear to be as many squares as there are integers; on the other hand, the totality of integers includes all the squares. Precisely because "the number of squares is not less than the totality of all numbers, nor the latter *greater than* the former," Galileo insists that "the attributes 'equal,' 'greater,' and 'less' are not applicable to infinite, but only to finite quantities." Nor does the sense in which one finite quantity can be greater or less than another—that is, by a determinate difference between them—apply in the comparison of a finite and an infinite quantity. The latter, being indeterminately large, is indeterminately larger than any finite quantity.

These remarks apply to the infinitely small as well. The infinitesimal is immeasurably small or indeterminately less than any finite fraction, no matter how small, because its own size is indeterminate. The finite fraction, itself a product of division, can be divided again, but if an infinitesimal quantity were capable of further division, it would permit a smaller, and since that smaller quantity would be a determinate fraction of itself, the infinitesimal would have to be determinate in size. Since that is not so, the infinitesimal must be conceived as the indivisible or as the limit approached by carrying on division endlessly.

"Because the hypothesis of indivisibles seems somewhat harsh," Newton proposes an analysis in terms of what he calls "nascent and evanescent quantities," or quantities just *beginning* to be more than nothing or just at the point at which they *vanish* into nothing. "As there is a limit which the velocity at the end of a motion may attain, but not exceed . . . there is a like limit in all quantities and proportions that begin or cease to be." Newton warns his reader, therefore, that if he "should happen to mention quantities as least, or evanescent, or ultimate," the reader is "not to suppose that quantities of any determinate magnitude are meant, but such as are conceived to be always diminished without end."

Later, speaking of quantities which are "variable and indetermined, and increasing or decreasing, as it were, by a continual motion or flux," he adds: "Take care not to look upon finite quantities as such." The method of fluxions provides an infinitesimal calculus on the hypothesis of limits rather than of indivisibles.

THROUGH ALL THESE conceptions of infinity—metaphysical, mathematical, and physical—run the paired notions of the unlimited and of limits approached but not attained. The finite is neither unlimited nor does it insensibly approach a limit. There are also the opposite notions of the perfect and the indeterminate. The finite is neither, for it is determinate without being a totality or complete.

Though they have a common thread of meaning, and though each raises similar difficulties for the understanding, the conception of infinity in being or power, and the conception of infinite (or infinitesimal) quantity require separate consideration. The same questions may be asked of each, questions about the existence of the infinite and about our knowledge of it, but the same answers will not be given in each case. There are those who deny the existence of an actually infinite body or an actually infinite number, yet affirm the infinite existence of God. There are those who declare the infinity of matter to be intrinsically unintelligible, but maintain that God, Who is infinite, is intrinsically the most intelligible object. They add, of course, that the infinite being of God cannot be comprehended by our finite intellects.

On each of these points, an opposite view has been taken, but the dispute concerning the infinity of God involves issues other than those which occur in the controversy over the infinite divisibility of matter or the infinity of space and time. It seems advisable, therefore, to deal separately with the problems of infinity as they arise with respect to different objects or occur in different subject matters.

THE CONCEPTION of God, in the words of Anselm, as a being "than which a greater cannot be conceived"—or, in the words of Kant, as an *ens realissimum*, a most real being—expresses the plenitude of the divine nature and existence. The medieval thesis, defended by Descartes, that God's essence and existence are identical, implies that neither is contracted or determined by the other. The still earlier notion of Aristotle, repeated by Aquinas, that God is pure actuality, carries with it the attributes of completeness or perfection, which are the positive aspects of immutability or incapacity for change. Spinoza's definition of substance as that which exists, not only in itself, but through itself and by its very nature, entails the autonomy or utter independence of the divine being.

These are so many different ways of stating that God is an infinite being. Both Aquinas and Spinoza make infinity the basis for proving that there can be only one God. When Spinoza argues that "a plurality of substances possessing the same nature is absurd," he has in mind the identification of infinite substance with God. "If many gods existed," Aquinas writes, "they would necessarily differ from each other. Something would therefore belong to one, which did not belong to another. And if this were a privation, one of them would not be absolutely perfect; but if it were a perfection, one of them would be without it. So it is impossible for many gods to exist"—that is, of course, if infinity is a property of the divine nature. Aquinas makes this condition clear when he goes on to say that "the ancient philosophers, constrained as it were by the truth, when they asserted an infinite principle, asserted likewise that there was only one such principle."

But while it is impossible for there to be two infinities of being, it is not impossible for there to be two, or more, infinite quantities. One explanation of this difference seems to be the actuality or existence of an infinite being, in contrast to the conceptual character of the infinite objects of mathematics, which are sometimes called "potential infinites" because they are conceived as in an endless process of becoming, or as approaching a limit that is never reached.

When the physical existence of infinite quantities is asserted, as, for example, a universe of infinite extent or an infinite number of atoms, the uniqueness of these actual totalities seems to follow. Two infinite worlds cannot coexist, though the one world can be infinite in several distinct respects—in space or duration, or in the number of its constituents—even as the infinity of God, according to Spinoza, involves "infinite attributes, each one of which expresses eternal and infinite essence."

Spinoza's argument against two actual infinities seems to find confirmation in the position taken by Aquinas that God's omnipotence does not include the power to create an infinite world. God's infinity, as we have already noted, follows from the identity of God's essence and existence. Since a created being has existence added to its essence, Aquinas asserts that "it is against the nature of a created thing to be absolutely infinite. Therefore," he continues, "as God, although He has infinite power, cannot make a thing to be not made (for this would imply that two contradictories are true at the same time), so likewise He cannot make anything to be absolutely *infinite*."

On this view, an infinite world cannot coexist with an infinite God, if, in their separate existence, one is dependent on the other, as creature upon creator. The infinity of the world or of nature, in Spinoza's conception, is not separate from the infinity of God, but consists in the infinity of two of God's attributes—extension and thought.

In our time there has arisen the conception of a finite God—a God who, while the most perfect being, yet is not without capacity for growth or change, a God who is eternal without being immutable. This conception, which in the light of traditional theology appears to be as self-contradictory as *round square*, has arisen in response to the difficulties certain critics have found in the traditional doctrine of an infinite being. They point to the difficulty of understanding how finite beings can exist separate from, yet in addition to, an infinite being; they also cite difficulties in the notions of infinite knowledge, infinite power, and infinite goodness.

The infinity of the divine omniscience extends to the possible as well as to the actual. But the possible includes things which are incompatible with one another, things which, in the language of Leibniz, are not *compossible*. The *incompossible* would thus seem to be embraced in the infinite scope of divine thought or knowledge. In the view of one theologian, Nicholas of Cusa, the mystery of God's infinity is best expressed by affirming that in God all contradictions are somehow reconciled.

The infinity of God's power, or the divine omnipotence, also raises questions about the possible and the impossible. Is nothing impossible to God or must it be said that there are certain things which not even God can do, such as reverse the order of time or create a world which shall be as infinite and perfect as himself? In the assertion that God cannot do the impossible, Aquinas sees no limitation on God's power. The impossible, he writes, does not "come under the divine omnipotence, not because of any defect in the power of God, but because it has not the nature of a feasible or possible thing." For this reason, he claims, "it is better to say that such things cannot be done, than that God cannot do them." The inability to do the *undoable* constitutes no violation of infinite power, even as the lack of nothing does not deprive infinite being of anything.

The infinite goodness of God is sometimes set against the fact of evil, or the existence of imperfections, in the created world. This aspect of the problem of evil, like that which concerns man's freedom to obey or disobey the divine will, cannot be separated from the fundamental mystery of God's infinity—in power and knowledge as well as in goodness. The problem is considered in the chapter on GOOD AND EVIL. The point there mentioned, that evil is essentially nonbeing or deprivation of being, leads to one solution of the problem. It accepts the finitude, and consequently the imperfection, of creatures as a necessary consequence of God's infinity. The best of all possible worlds cannot be infinitely good.

TO MAN ALONE, among all admittedly finite things, has infinity been attributed and even made a distinctive mark of his nature. Does this introduce a new meaning of infinity, neither quantitative nor divine?

It has seldom if ever been questioned that man is finite in being and power. The limits of human capacity for knowledge or achievement are a perennial theme in man's study of man. Yet it is precisely with regard to *capacity* that certain writers have intimated man's infinity.

Pascal, for example, finds the apparent contradictions in human nature intelligible only when man is understood as yearning for or impelled toward the infinite. "We burn with desire," he says, "to find solid ground and an ultimate sure foundation whereon to build a tower reaching to the Infinite. But our whole groundwork cracks and the earth opens to abysses." In this fact lies both the grandeur and the misery of man. He aspires to the infinite, yet he is a finite being dissatisfied with his own finitude and frustrated by it.

It is sometimes said that the touch of infinity in man—with the suggestion that it is a touch of madness—consists in his wanting to be God. Those who regard such desire as abnormal or perverse interpret it as a misdirection of man's natural desire to know God face to face and to be filled with the love of God in the divine presence. But, according to the theory of natural desire, the tendency of each nature is somehow proportionate to its capacity. If man's restless search for knowledge and happiness can be quieted only by the possession of the infinite truth and goodness which is God, then man's intellect and will must somehow be as infinite in nature as they are in tendency. Yet that is not an unqualified infinity, for the same theologians who teach that man naturally seeks God also hold that man's finite intellect cannot *comprehend* the infinite being of God as God knows Himself. Nor do they think that man's capacity for knowing and loving God can be fulfilled except in the beatific vision, which is a supernatural gift rather than a natural achievement.

These and related matters are discussed in the chapters on DESIRE and KNOWLEDGE. The great books speak of other objects than God as objects of man's infinite desire. The appetite for money, for pleasure, or for power seems to be an infinite craving which no finite quantity of these goods ever satisfies. Two comments are made upon this fact, which is so amply evidenced in the human record. One is that man's infinite lust for worldly goods expresses even as it conceals his natural desire for a truly infinite good. The other is that these worldly goods are seductive objects precisely because they are infinite.

Here the word "infinite" is used, not in the sense which signifies perfection, but in the quantitative sense which has the meaning of indetermination. Plato's division, in the *Philebus,* of goods into the finite and the infinite separates measured and definite goods from those which need some limitation in quantity. Socrates exemplifies the distinction by reference to the fact that "into the hotter and the colder there enters a more and a less" and since "there is never any end of them . . . they must also be infinite." In contrast, "when definite quantity is once admitted, there can be no longer a 'hotter' or a 'colder.' " Such things, he says, "which do not admit of more or less" belong "in the class of the limited or finite."

Following the line of this example, Socrates later distinguishes between infinite and finite pleasures, or pleasures without limit and those which have some intrinsic measure. "Pleasures which are in excess," he says, "have no measure, but those which are not in excess have measure; the great, the excessive . . . we shall be right in referring to the class of the infinite, and of the more and less," and "the others we shall refer to the class which has measure." The fact that the goodness of wealth or of certain pleasures is indeterminate or indefinite makes it necessary to determine or measure the amount of wealth it is good to possess, or the quantity of such pleasure it is good to enjoy.

As in the case of desire, so the human intellect is also said to be infinite in the sense of reaching to an indefinite quantity. On the theory which he holds that the intellect knows by means of universal concepts, Aquinas attributes to the human mind "an infinite power; for it apprehends the universal, which can extend itself to an infinitude of singular things." Each universal signifies what is common to an indefinitely large class of particular instances.

There is still another sense in which the intellect is said to be infinite, namely, by reason of its having the potentiality to apprehend *all* knowable things. But this is a relative infinity, as is the corresponding infinity of prime matter, which is conceived as the potentiality for taking on all forms. In both cases, the infinite is qualified by a restriction—on the kind of things knowable to the intellect and the type of forms receivable in matter. The infinity of prime matter—matter totally devoid of form—is also comparable to the infinity of God in a contrast of extreme opposites: the absolute indeterminacy of pure potentiality on the one hand, the absolute perfection of pure actuality on the other.

THE INFINITY OF matter involves different considerations when the problem concerns, not prime matter, but material things—bodies. The question is twofold. Can there be a body of infinite magnitude? Is there an infinite number of bodies? To both questions Aristotle gives the negative answer, while Spinoza seems to answer the first, and Lucretius the second, affirmatively.

Spinoza's affirmation may be qualified, of course, by his conception of infinite body as an attribute of God. But there is no qualification on Lucretius' assertion that "there must be an infinite supply of matter," unless it is his statement that "atoms have a finite number of differing shapes." It is only the number of atoms which is infinite, not their variety.

Aristotle presents many arguments against the existence of an infinite body or an infinite number of things, all of which ultimately rest on his distinction between an actual and a potential infinite. It is not that infinity in magnitude or multitude is impossible—for he affirms the infinity of time and he insists upon the infinite divisibility of matter—but rather that if an infinite body existed its infinity would have to be actual. Its actuality would necessarily involve certain determinations, especially those of dimension and place, which would be inconsistent with the indeterminacy of the infinite. Similarly, a multitude of coexisting things—unlike the moments of time which do not coexist—cannot be infinite, because their coexistence implies that they can be actually numbered, whereas their infinity implies that they are numberless.

The potential infinite, Aristotle writes, "exhibits itself in different ways—in time, in the generations of man, and in the division of magnitudes. For generally," he says, "the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different." When this takes place in the division of spatial magnitudes, "what is taken persists, while in the succession of times and of men, it takes place by the passing away of these in such a way that the source of supply never gives out."

The opposition between Lucretius and Aristotle with regard to the divisibility of matter is discussed in the chapter on ELEMENT. The notions of infinity and continuity are differently employed on the two sides of the argument. Where Aristotle makes the continuity of matter the condition of its infinite divisibility, Lucretius makes the atom's continuity—its solidity or lack of void—the cause of its indivisibility. Where Aristotle asserts that at any moment there can be only a finite number of particles in the world because the partition of matter cannot be infinitely carried out short of infinite time, Lucretius, on the contrary, thinks that the division of matter into smaller and smaller parts finds an end in the atomic particles; and yet he also asserts an infinite number of atoms.

To contain an infinite number of atoms, an infinite space is required, according to Lucretius. This presents no greater difficulty for him than an infinite time. Aristotle, on the other hand, differentiates between space and time with respect to infinity. Time can be potentially infinite by way of addition because "each part that is taken passes in succession out of existence." But though space may be infinitely divisible, it cannot be infinitely extended, for all its parts, unlike those of time, must coexist. It would therefore have to be an *actually,* rather than a *potentially,* infinite quantity, and this Aristotle thinks is impossible.

These and other conflicting views concerning the infinity of space and time appear in Kant's statement of the first cosmological antinomy. His intention is not to resolve the issues, but to show that they cannot be resolved by proof or argument. To do this, Kant sets up what seems to him to be equally strong—or equally inconclusive—arguments for and against the infinity of space and time.

Suppose it be granted, Kant argues on the one hand, that "the world has no beginning in time." Then it would follow that "up to every given moment in time, an eternity must have elapsed, and therewith passed away an infinite series of successive conditions or states of things in the world." But since "the infinity of a series consists in the fact that it can never be completed by means of a successive synthesis," it also "follows that an infinite series already elapsed is impossible, and that consequently a beginning of the world is a necessary condition of its existence."

On the other hand, Kant argues with what he thinks is equal force, "let it be granted that [the world] has a beginning. A beginning," he explains, "is an existence which is preceded by a time in which the thing does not exist." Then, Kant continues, "on the above supposition, it follows that there must have been a time in which the world did not exist, that is, a void time. But in a void time, the origination of a thing is impossible; because no part of any such time contains a distinctive condition of being in preference to that of non-being . . . Consequently, many series of things may have a beginning in the world, but the world itself cannot have a beginning, and is, therefore, in relation to past time, infinite."

With regard to the infinity or finitude of space, Kant proceeds similarly. If we suppose space to be infinite, then "the world must be an infinite given total of co-existent things." But in order to "cogitate the world, which fills all space as a whole, the successive synthesis of the parts of an infinite world must be looked upon as completed; that is to say; an infinite time must be regarded as having elapsed in the enumeration of all co-existing things." This, Kant argues, "is impossible," and therefore "an infinite aggregate of actual things cannot be considered as a given whole." Hence it follows that "the world is, as regards extension in space, *not infinite,* but enclosed in limits."

If, however, we suppose "that the world is finite and limited in space, it follows," according to Kant, "that it must exist in a void space, which is not limited. We should, therefore, meet not only with a relation of things *in space,* but also a relation of things *to space.*" But the "relation of the world to a void space is merely a relation to *no object*" and "such a relation, and consequently the limitation of the world by void space, is nothing." It follows, therefore, Kant concludes, that "the world, as regards space, is not limited; that is, it is infinite in regard to extension."

The way in which these opposite arguments nullify each other reveals more than our inability to prove or disprove the infinity of space and time. It shows, in Kant's theory of human knowledge, that we are "not entitled to make any assertion at all respecting the whole object of experience—the world of sense."

"Space and time," Russell writes, "appear to be infinitely divisible. But as against these apparent facts—infinite extent and infinite divisibility—philosophers have advanced arguments tending to show that there could be no infinite collections of things, and that therefore the number of points in space, or of instants in time, must be finite. Thus a contradiction emerged between the apparent nature of space and time and the supposed impossibility of infinite collections."

One other problem of infinity in the sphere of physics receives its initial formulation in one of the great books—in the part of the *Dialogues Concerning the Two New Sciences* where Galileo discusses the uniform acceleration of a freely falling body. The body which is said to accumulate equal increments of velocity in equal intervals of time is also said to start "from infinite slowness, *i.e.*, from rest." One of the persons in the dialogue challenges this, saying that "as the instant of starting is more and more nearly approached, the body moves so slowly that, if it kept on moving at this rate, it would not traverse a mile in an hour, or in a day, or in a year, or in a thousand years; indeed, it would not traverse a span in an even greater time; a phenomenon which baffles the imagination, while our senses show us that a heavy falling body suddenly acquires great speed."

What our senses *seem* to show us is corrected by an experiment which refines the observation. But this still leaves a purely analytical question. Against the statement that the "velocity can be increased or diminished without limit," Simplicio points out in the dialogue that "if the number of degrees of greater and greater slowness is limitless, they will never be all exhausted," and therefore the body will never come to rest when it is slowing down or be able to start to move when it is at rest.

"This would happen," Salviati answers, "if the moving body were to maintain its speed for any length of time at each degree of velocity, but it merely passes each point without delaying more than an instant, and since each time interval, however small, may be divided into an infinite number of instants, these will always be sufficient to correspond to the infinite degrees of diminished velocity."

The problem of the infinitesimal velocity provides another illustration of the difference between infinity in the physical and the mathematical orders. Unlike parallel lines in Euclidean geometry, which are lines that remain equidistant from one another when both are prolonged to infinity, an asymptote is a straight line which a curved line continuously approaches but never meets, even when both are infinitely extended. The distance between the curve and its asymptote diminishes to smaller and smaller intervals, but no matter how small they become, the two lines never coincide. The diminishing intervals between the curve and its asymptote are like the diminishing degrees of velocity in a body starting from or coming to rest. But we know that the body does begin or cease to move, and so there is the mysterious jumping of the gap between rest and motion in the physical order, whereas in the mathematical order the limiting point can be forever approached and never reached.

THERE IS ONE other context in which infinity is discussed in the great books.

The logicians treat certain terms and judgments as infinite. Aristotle, for example, regards the negative term—such as *not-man* or *not-white*—as indefinite. The indefiniteness of its signification may be seen when such terms are used as subjects of discourse. What is being talked about? The answer must be given, in part at least, in positive terms: *not-man* represents the *whole universe* leaving man out, or the *totality of everything* except man. Thus, in its positive signification, the negative term has a kind of infinity—the infinite totality of subjects diminished by one, the one that is negated.

In his classification of judgments, Kant makes a threefold division of judgments according to quality: the affirmative, the negative, and the infinite. The infinite judgment involves a negative in its construction, but when that negative is given an affirmative interpretation, the infinite significance of the proposition becomes apparent. An example will make this clear.

The proposition *this animal is-not white* is negative; it simply denies a certain quality of a certain thing. But the proposition *this animal is not-white* is infinite, for it affirms the negated term, and so places the subject in the infinite class or totality which includes everything except white things. (The position of the hyphen serves to indicate whether the statement shall be construed negatively or affirmatively *and* infinitely.)

The problems of definition and demonstration are differently solved by logicians according to the way in which they propose to avoid infinite regressions in analysis or reasoning. There would be no end to the process of defining if every term had to be defined before it could be used in the definition of another term. There would be no beginning to the process of proof if, before a proposition could be used as a premise to demonstrate some conclusion, it had itself to be demonstrated as a conclusion from prior premises.

In his essay "On Geometrical Demonstration," Pascal refers to the proposal of a plan for defining and proving everything. "Certainly this method would be beautiful," he says, "but it is absolutely impossible; for it is evident that the first terms we wished to define would presuppose others for their explication, and that similarly the first propositions we wished to prove would suppose others that preceded them, and it is thus clear we should never arrive at the first propositions."

The chapter on DEFINITION considers the character and choice of the indefinable terms by which an infinite regression is avoided in the elucidation of meanings. The chapters on INDUCTION and PRINCIPLE consider the various sorts of primary propositions—axioms, postulates, assumptions—by which a similar regression is avoided in the process of proof. The chapter on CAUSE deals with the problem of an infinite regression in causes and effects. Here it is appropriate to consider the difference between an infinite series of reasons and an infinite series of causes.

To the extent that both are truly series—the succession of one thing after another—neither seems to be impossible, *given infinite time*. Those who deny the possibility of an infinite number of causes distinguish between essential and accidental causes, that is, between causes which must coexist with their effects and causes which can precede their effects, and cease to be before their effects occur. If there were an infinite time, there could be an infinite series of accidental causes. But it may be questioned whether, even granted an infinite time, the relation between the premises and conclusion of reasoning permits an infinite regression. If the truth of a conclusion cannot be known until the truth of its premises is known, then the pursuit of truth may be vitiated by a search *ad infinitum.*

AT THE END OF THE 19th century, especially with the work of Georg Cantor, new insights into the nature of infinity in mathematics emerged. The number of objects in a set of objects is equal to the number of objects in a second set if the objects in each set can be paired off with each other in a one-to-one correspondence. Hence the fingers on the left hand and the fingers on the right hand can be so paired, meaning that each hand has an identical number of fingers.

By definition, a set with an infinite number of elements is a set that has the same number of elements as one of its subsets. Thus the number of positive integers is infinite because the positive integers can be paired off in a one-to-one correspondence with, for example, the even integers. This number is, in the notation of Cantor, designated as "aleph naught." The great discovery of Cantor was that this is the smallest of the infinite numbers.

A larger infinite number is the number of points on a line—a number which is called "C," for the number of points in a continuum. Indeed, there is an entire hierarchy of transfinite numbers. When this discovery was first made, it was considered extremely paradoxical. Now, it is a standard part of contemporary mathematics.